Estimation on the Convergence of the Semi-Exponential Szász-Mirakyan-Kantorovich Operators

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Abstract: The approximation rate of some Szász-Mirakyan-Kantorovich operators for some absolutely continuous functions is obtained. Firstly, the author introduces the new Szász-Mirakyan operators of the integral form. After a simple calculation, the first and second order central moment of the new integral form operators are given. Late, using analysis techniques the approximation theorem of the new operators is decomposed into several parts. Then, each estimate is calculated. Lastly, an asymptotically optimal estimate is obtained as same as the classical method of Bojanic and Cheng. The conclusion drawn in this article extends the research findings of Agrawal and Gupta.

1. Introduction

Weierstrass theorem tells us that any continuous function can be approximated by infinitely differentiable polynomial functions. There are many ways to prove this theorem, among which the most famous one belongs to Bernstein's proof. He constructed an n-order polynomial, also known as the Bernstein operator, to prove it. On this basis, many scholars have discussed other types of operators. In order to study approximations on infinite intervals, Szász defined operators $S_n(\varphi,\chi)=\sum_{j=0}^{\infty}\varphi(\frac{j}{n})s_{nj}(\chi)$, $s_{nj}(\chi)=e^{-n\chi}\frac{(n\chi)^j}{j!},\chi\in[0,+\infty)$.

Herzog[1,2] introduced the semi-exponential Kantorovich variant of S_n defined by

$$S_{n,\lambda}(\varphi,\chi)=\sum_{j=0}^{\infty}\varphi(\frac{j}{n})s_{nj}^{\lambda}(\chi),\lambda\geq 0 \tag{1}$$

where $s_{nj}^{\lambda}(\chi)=e^{-(n+\lambda)\chi}\frac{(n+\lambda)^j\chi^j}{j!},\chi\in[0,+\infty)$. If $\lambda=0$, $S_{n,\lambda}$ change into the classical Szász-Mirakjan operators.

Herzog[1,2] studied the approximation properties of $S_{n,\lambda}$ in exponential weighted spaces. The author focused on discussing modified operators and derived a new operators. The probability methods for these modifications were also demonstrated. The author concentrated one's attention on the modification of $S_{n,\lambda}$ in another variant, and examined the approximation rate of basic operators and modified operators. The proposed reasoning route emphasized some symmetries in the modification of $S_{n,\lambda}$.

Based on the research of Herzog，Agrawal et al [3] introduced the Kantorovich type of $S_{n,\lambda}$ as follows:

$$K_{n,\lambda}(\varphi,\chi)=n\sum_{j=0}^{\infty}s_{nj}^{\lambda}(\chi)\int_{j/n}^{(j+1)/n}\varphi(t)\mathrm{d}t. \tag{2}$$

In [3], the approximation rate of operators $K_{n,\lambda}$ to bounded variation functions was studied.

Also, the voronovskaya type asymptotic expression was obtained.

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The rate of approximation for $DBV[0,\infty)$ is a research hotspot. The research in this field started from Bojanic and Cheng [4]. Subsequently, A series of achievements have emerged in this field one after another [5-9]. Afflected by this, the paper studies the approximation of operator $K_{n,\lambda}(\varphi, y)$ for $DBV[0,\infty)$ which is defined as follows.

For the convenience of studying the problem, we first provide some definitions.

Definition 1 $DBV[0,\infty)=\left\{\varphi \middle| \varphi(\chi)=\varphi(0)+\int_0^{\chi}\ell(z)dz\right\}$

where $\chi\in[0,\infty), \ell\in BV[0,\infty)$.

Let $\Im_j(t)$ be the characteristic function which is defined in the interval of $[j/n,(j+1)/n]$, we provide the definition of kernel functions $\vartheta_n(\chi,t)$.

Definition 2 $\vartheta_n(\chi,t)=n\sum_{j=0}^{\infty}s_{nj}^{\lambda}(\chi)\Im_j(t)$.

Using Definition 2 we have

$$K_{n,\lambda}(\varphi,\chi)=\int_0^{\infty}\varphi(t)\vartheta_n(\chi,t)\mathrm{d}t. \tag{3}$$

2. Preliminary Lemmas

Some basic conclusions for proof of the conclusion are needed.

Lemma 1(see [3]) If $\chi\in[0,+\infty)$, then we have

$$K_{n,\lambda}(1,\chi)=1, \tag{4}$$

$$K_{n,\lambda}(z,\chi)=\chi+\frac{2\lambda\chi+1}{2n}, \tag{5}$$

$$K_{n,\lambda}(z^2,\chi)=\chi^2+\frac{3\lambda^2\chi^2+6\lambda\chi(n\chi+1)+6n\chi+1}{3n^2}. \tag{6}$$

$$K_{n,\lambda}(z-\chi,\chi)=\frac{2\lambda\chi+1}{2n}, \tag{7}$$

$$K_{n,\lambda}\left((z-\chi)^2,\chi\right)=\frac{3\lambda^2\chi^2+6\lambda\chi+3n\chi+1}{3n^2}\triangleq\Psi(n,\lambda,\chi). \tag{8}$$

Lemma 2 If $\chi\in[0,+\infty)$, then we have

$$K_{n,\lambda}(|t-\chi|,\chi)\le\sqrt{\Psi(n,\lambda,\chi)}. \tag{9}$$

Proof. According to the famous Cauchy-Schwarz inequality, we can immediately obtain

$$K_{n,\lambda}\left(|t-\chi|,y\right)\le\sqrt{K_{n,\lambda}\left((t-\chi)^2,\chi\right)}\cdot\sqrt{K_{n,\lambda}(1,\chi)}=\sqrt{\Psi(n,\lambda,\chi)}.$$

The last equality is obtained by (4) and (8).

Lemma 3 (i) Supposing that $0<t<\chi<+\infty$, so

$$\gamma_n(\chi,t)=\int_0^t\vartheta_n(\chi,u)\mathrm{d}u\le\frac{\Psi(n,\lambda,\chi)}{(t-\chi)^2}.$$

(ii) Supposing that $0<z<t<+\infty$, so

$$1-\gamma_n(z,t)=\int_t^{+\infty}\vartheta_n(z,u)\mathrm{d}u\le\frac{\Psi(n,\lambda,z)}{(t-z)^2}.$$

Proof. (i) By the expression of (3) and (8), we can derive

$$\gamma_n(\chi,t)=\int_0^t \vartheta_n(\chi,u)\mathrm{d}u \le \int_0^t (\frac{\chi-u}{t-\chi})^2 \vartheta_n(\chi,u)\mathrm{d}u \le \frac{1}{(t-\chi)^2}\int_0^{+\infty}(\chi-u)^2\vartheta_n(\chi,u)\mathrm{d}u$$

$$=\frac{1}{(t-\chi)^2}K_{n,\lambda}\left((u-\chi)^2,\chi\right)=\frac{\Psi(n,\lambda,\chi)}{(t-\chi)^2}.$$

(ii) A completely similar method can be used to obtain (ii) easily.

3. Conclusion

Theorem 1 Let $\varphi\in DBV[0,\infty)$ and $\varphi(z)=O(z^{\alpha z})(z\to\infty,\alpha>0)$. If $\ell(\chi+),\ell(\chi-)$ exist at a fixed point $\chi\in(0,\infty)$, when n sufficient large we have

$$\left|K_{n,\lambda}(\varphi,\chi)-\varphi(\chi)-\frac{2\lambda\chi+1}{4n}\left[\ell(\chi+)+\ell(\chi-)\right]\right|$$

$$\le \frac{\left|\ell(\chi+)-\ell(\chi-)\right|}{2}\sqrt{\Psi(n,\lambda,\chi)}\ +O(1)\frac{(2\chi+1)^{(2\chi+1)\alpha}}{1+\sqrt{(n+\lambda)\chi}}(\frac{e}{4})^{(n+\lambda)\chi}$$

$$+\Psi(n,\lambda,\chi)\cdot\frac{2}{\chi}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{\chi-\frac{\chi}{k}}^{\chi+\frac{\chi}{k}}(\eta_\chi)+\frac{\chi}{\sqrt{n}}\bigvee_{\chi-\frac{\chi}{\sqrt{n}}}^{\chi+\frac{\chi}{\sqrt{n}}}(\eta_\chi),$$

where $\eta_y(t)=\begin{cases}\ell(t)-\ell(\chi+), \chi<t<\infty\\ 0, \qquad t=\chi\\ \ell(t)-\ell(\chi-), 0\le t<\chi\end{cases}$.

Proof. By the definition of functions φ and the method of reference [4], we get

$$\varphi(t)-\varphi(\chi)=\int_\chi^t \ell(u)du \tag{10}$$

where

$$\ell(u)=\frac{\ell(\chi+)+\ell(\chi-)}{2}+\eta_\chi(u)+\frac{\ell(\chi+)-\ell(\chi-)}{2}sign\left(u-\chi\right)+\varpi_\chi(u)\left[\ell(\chi)-\frac{\ell(\chi+)+\ell(\chi-)}{2}\right], \tag{11}$$

and $\varpi_\chi(u)=\begin{cases}1 \ , \ u=\chi\\ 0 \ , \ u\ne\chi\end{cases}$, $sign(\chi)=\begin{cases}1, & \chi>0;\\ 0, & \chi=0;\\ -1, & \chi<0.\end{cases}$

From (10) and (11), we have

$$\left|K_{n,\lambda}(\varphi,\chi)-\varphi(\chi)-\frac{\ell(\chi+)+\ell(\chi-)}{2}K_{n,\lambda}(t-\chi,\chi)\right|$$

$$=\left|\frac{\ell(\chi+)-\ell(\chi-)}{2}K_{n,\lambda}(|t-\chi|,\chi)+K_{n,\lambda}(\int_\chi^t\eta_\chi(u)\mathrm{d}u,\chi)\right|.$$

By (7) and (9), we have

$$\left|K_{n,\lambda}(\varphi,\chi)-\varphi(\chi)-\frac{2\lambda\chi+1}{4n}\left[\ell(\chi+)+\ell(\chi-)\right]\right|$$

$$\le \frac{|\ell(\chi+)-\ell(\chi-)|}{2}\sqrt{\Psi(n,\lambda,\chi)}+\left|K_{n,\lambda}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u,\chi)\right|. \tag{12}$$

Next, we estimate the term $K_{n,\lambda}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u,\chi)$.

From (3) and Lemma 3, the term $K_{n,\lambda}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u,\chi)$ can be expressed as

$$K_{n,\lambda}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u,\chi)=\int_{0}^{\infty}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u)\mathrm{d}_{t}\gamma_{n}(\chi,t)=\Pi_1+\Pi_2+\Pi_3, \tag{13}$$

where

$$\Pi_1=\int_{0}^{\chi}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u)\mathrm{d}_{t}\gamma_{n}(\chi,t),\Pi_2=\int_{\chi}^{2\chi}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u)\mathrm{d}_{t}\gamma_{n}(\chi,t),\Pi_3=\int_{2\chi}^{\infty}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u)\mathrm{d}_{t}\gamma_{n}(\chi,t).$$

Firstly, we estimate Π_1. As is known to all that $\gamma_n(\chi,0)=0,\int_{\chi}^{\chi}\eta_{\chi}(u)\mathrm{d}u=0$. According to integration by parts, we get

$$\Pi_1=\int_{0}^{\chi}(\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u)\mathrm{d}_{t}\gamma_{n}(\chi,t)=\gamma_n(\chi,t)\int_{\chi}^{t}\eta_{\chi}(u)\mathrm{d}u\Big|_0^{\chi}-\int_0^{\chi}\gamma_n(\chi,t)\eta_{\chi}(t)\mathrm{d}t=-\int_0^{\chi}\gamma_n(\chi,t)\eta_{\chi}(t)\mathrm{d}t$$

$$=-\left(\int_0^{\chi-\frac{\chi}{\sqrt{n}}}+\int_{\chi-\frac{\chi}{\sqrt{n}}}^{\chi}\right)\gamma_n(\chi,t)\eta_{\chi}(t)\mathrm{d}t.$$

Thus,

$$|\Pi_1|\le\int_0^{\chi-\frac{\chi}{\sqrt{n}}}\gamma_n(\chi,t)\mathop{V}_{t}^{y}(\eta_{\chi})\mathrm{d}t+\int_{\chi-\frac{\chi}{\sqrt{n}}}^{\chi}\gamma_n(\chi,t)\mathop{V}_{t}^{y}(\eta_{\chi})\mathrm{d}t.$$

By the definition of $\gamma_n(\chi,t)$ and noting $0\le\gamma_n(\chi,t)\le1$, we have

$$|\Pi_1|\le\Psi(n,\lambda,\chi)\cdot\int_0^{\chi-\frac{\chi}{\sqrt{n}}}\frac{\mathop{V}_{t}^{\chi}(\eta_{\chi})}{(\chi-t)^2}\mathrm{d}t+\frac{\chi}{\sqrt{n}}\mathop{V}_{\chi-\frac{\chi}{\sqrt{n}}}^{\chi}(\eta_{\chi}). \tag{14}$$

Putting $t=\chi(1-\frac{1}{u})$ for the integral of (14), we get

$$\int_0^{\chi-\frac{\chi}{\sqrt{n}}}\frac{\mathop{V}_{t}^{\chi}(\eta_{\chi})}{(\chi-t)^2}\mathrm{d}t=\frac{1}{\chi}\int_1^{\sqrt{n}}\mathop{V}_{\chi-\frac{\chi}{u}}^{\chi}(\eta_{\chi})\mathrm{d}u\le\frac{2}{\chi}\sum_{k=1}^{[\sqrt{n}]}\mathop{V}_{\chi-\frac{\chi}{k}}^{\chi}(\eta_{\chi}). \tag{15}$$

From (14) and (15), it follows that

$$|\Pi_1|\le\Psi(n,\lambda,\chi)\cdot\frac{2}{\chi}\sum_{k=1}^{[\sqrt{n}]}\mathop{V}_{\chi-\frac{\chi}{k}}^{\chi}(\eta_{\chi})+\frac{\chi}{\sqrt{n}}\mathop{V}_{\chi-\frac{\chi}{\sqrt{n}}}^{\chi}(\eta_{\chi}). \tag{16}$$

A completely similar method can be used to obtain the following inequality

$$|\Pi_2|\le\Psi(n,\lambda,\chi)\cdot\frac{2}{\chi}\sum_{k=1}^{[\sqrt{n}]}\mathop{V}_{\chi}^{\chi+\frac{\chi}{k}}(\eta_{\chi})+\frac{\chi}{\sqrt{n}}\mathop{V}_{\chi}^{\chi+\frac{\chi}{\sqrt{n}}}(\eta_{\chi}). \tag{17}$$

Assumption by theorem that $\varphi(z)=O(z^{\alpha z})(z\to\infty,\alpha>0)$, we can the estimation of $|\Pi_3|$ by derivation of reference [10].

$$\left|\Pi_3\right| = O(1)\frac{(2\chi+1)^{(2\chi+1)\alpha}}{1+\sqrt{(n+\lambda)\chi}}(\frac{e}{4})^{(n+\lambda)\chi}. \tag{18}$$

Theorem now follows from (12), (13), (16), (17) and (18).

Acknowledgements

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